# Module-1: Contour Integration-I

## 1 Introduction

A variety of real definite integrals can be evaluated with the help of Cauchy's residue theorem. In evaluating a real integral by contour integration, an appropriate complex function and an appropriate contour must be chosen. Here we illustrate the methods together with a suitable function $f$ and a suitable closed contour $C$; the choice, nevertheless, depends on the problem.

**Integration around the unit circle**

An integral of the form

$$\int_0^{2\pi} f(\sin\ \theta, \cos\ \theta) d\theta \tag{1}$$

where the integrand is a rational function of $sin\ \theta$ and $\cos\ \theta$, can be evaluated by putting $z = e^{i\theta}$. Since

$$\sin\ \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos\ \theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

the integral (1) takes the form

$$\int_C f(z) dz$$

where $f$ is a rational function of $z$ and $C$ is the unit circle $\mid z \mid = 1$.

**Example 1.** *Show that* $\int_0^{2\pi} \frac{d\theta}{5+3\cos\ \theta} = \pi/2$.

**Solution.** *Let* $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\ \theta}$. *We put* $z = e^{i\theta}$. *Then*

$$\cos\ \theta = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

*Therefore,*

$$\begin{aligned} I &= \int_{|z|=1} \frac{1}{5+\frac{3}{2}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz} \\ &= \frac{2}{i}\int_{|z|=1} \frac{dz}{3z^2+10z+3} \\ &= \frac{2}{i}\int_{|z|=1} \frac{dz}{(3z+1)(z+3)} \\ &= \frac{2}{i}\int_{|z|=1} f(z)dz, \end{aligned}$$

*where* $f(z) = \frac{1}{(3z+1)(z+3)}$. *Now* $f(z)$ *has simple poles at* $z = -1/3,\ -3$ *of which* $z = -1/3$ *lies inside the circle* $|z| = 1$. *Now*

$$Res(f;-1/3) = \lim_{z\to -1/3}(z+1/3)f(z) = \lim_{z\to -1/3}\frac{1}{3(z+3)} = \frac{1}{8}.$$

*Therefore by Cauchy's residue theorem we have*

$$I = \frac{2}{i}\cdot 2\pi i \cdot \frac{1}{8} = \frac{\pi}{2}.$$

*This completes the solution.*

**Example 2.** *Show that* $\int_0^{2\pi} \frac{\cos\ 2\theta}{1-2a\cos\ \theta+a^2}d\theta = \frac{2\pi a^2}{1-a^2},\ (a^2 < 1)$.

**Solution.** *Let* $I = \int_0^{2\pi} \frac{\cos\ 2\theta}{1-2a\cos\ \theta+a^2}d\theta$. *We put* $z = e^{i\theta}$. *Then*

$$\cos\ \theta = \frac{1}{2}\left(z+\frac{1}{z}\right),\quad \cos\ 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\ \text{ and }\ dz = ie^{i\theta}d\theta.$$

*Therefore,*

$$\begin{aligned} I &= \int_{|z|=1} \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)}{1-a\left(z+\frac{1}{z}\right)+a^2}\frac{dz}{iz} \\ &= \frac{1}{2i}\int_{|z|=1} \frac{z^4+1}{z^2(z-a)(1-az)}dz \\ &= \frac{1}{2i}\int_{|z|=1} f(z)dz, \end{aligned}$$

*where* $f(z) = \frac{z^4+1}{z^2(z-a)(1-az)}$. *Now* $f(z)$ *has simple poles at* $z = a,\ 1/a$ *and a pole of multiplicity 2 at* $z = 0$, *of which the poles at* $z = 0$ *and* $z = a$ *lie inside the circle* $|z| = 1$. *Now*

$$\begin{aligned} Res(f;a) &= \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\frac{z^4+1}{z^2(1-az)} \\ &= \frac{a^4+1}{a^2(1-a^2)}. \\ Res(f;0) &= \lim_{z\to 0}\frac{d}{dz}[z^2f(z)] = \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+1}{(z-a)(1-az)}\right] \\ &= -\frac{a^2+1}{a^2}. \end{aligned}$$

*Therefore by Cauchy's residue theorem we have*

$$I = 2\pi i \cdot \frac{1}{2i}\left[\frac{a^4+1}{a^2(1-a^2)} - \frac{a^2+1}{a^2}\right] = \frac{2\pi a^2}{1-a^2}.$$

*This completes the solution.*

**Integration around a semi-circle**

To evaluate $\int_{-\infty}^{\infty} f(x)dx$, we consider $\int_C f(z)dz$ where $C$ is the contour consisting of the semi-circle $C_R : |z| = R$ $(Im\ z \geq 0)$ together with the diameter that encloses it. Assuming that the function $f(z)$ has no singularities on the real axis, by Cauchy's residue theorem, we have

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_a Res(f;a).$$

Then proceeding to the limit as $R \to \infty$, we find the value of the integral $\int_{-\infty}^{\infty} f(x)dx$, provided $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

**Example 3.** *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx$.

**Solution.** *We consider*

$$\int_C \frac{z^2}{(z^2+1)(z^2+4)}dz = \int_C f(z)dz, \ say, \tag{2}$$

*where $C$ is the contour consisting of the semi-circle $C_R$ of radius $R$ together with the part of the real axis from $-R$ to $R$. The function $f(z)$ has simple poles at $z = \pm i$, $z = \pm 2i$ of which $z = i$ and $z = 2i$ lie inside the closed contour $C$ (see Fig 1.1). Hence by Cauchy's*

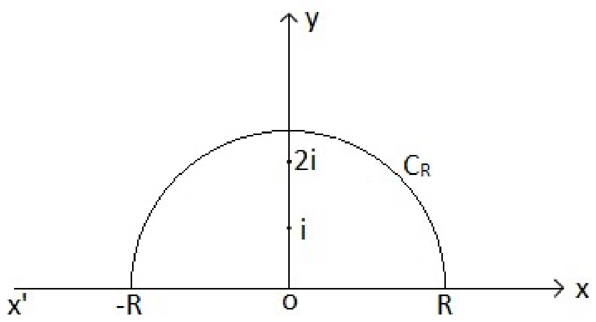


Fig. 1:

*residue theorem we obtain*

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i[Res(f;i) + Res(f;2i)]. \tag{3}$$

*Now*

$$\begin{aligned} Res(f;i) &= \lim_{z\to i}(z-i)f(z) \\ &= \lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)} = -\frac{1}{6i}. \end{aligned}$$

$$\begin{aligned} Res(f;2i) &= \lim_{z\to 2i}(z-2i)f(z) \\ &= \lim_{z\to 2i}\frac{z^2}{(z+2i)(z^2+1)} = \frac{1}{3i}. \end{aligned}$$

*Therefore from (3) we obtain*

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = \pi/3. \tag{4}$$

*Now on $C_R$,*

$$\begin{aligned} \mid f(z) \mid &= \frac{\mid z \mid^2}{\mid 1+z^2 \mid\mid 4+z^2 \mid} \leq \frac{\mid z \mid^2}{(\mid z \mid^2 -1)(\mid z \mid^2 -4)} \\ &= \frac{R^2}{(R^2-1)(R^2-4)} = \frac{1}{R^2}\frac{1}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})}. \end{aligned}$$

*Applying ML-formula we obtain*

$$\mid \int_{C_R} f(z)dz \mid \leq \frac{1}{R^2}\frac{1}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})} \cdot \pi R \to 0 \ as \ R \to \infty$$

$$i.e. \ \lim_{R\to\infty}\int_{C_R} f(z)dz = 0.$$

*Therefore, proceeding limit as $R \to \infty$ we obtain from (4)*

$$\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx = \pi/3.$$

*This completes the solution.*

**Example 4.** *Evaluate* $\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)}$.

**Solution.** *We consider*

$$\int_C \frac{1}{(z^2+1)^2(z^2+4)}dz = \int_C f(z)dz, \ say, \tag{5}$$

Now,

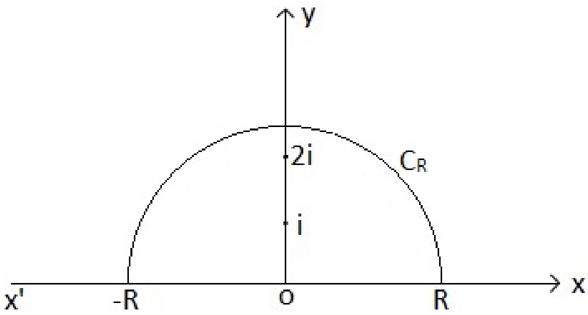


Fig. 2:

*where $C$ is the contour consisting of the semi-circle $C_R$ of radius $R\ (>2)$ together with the part of the real axis from $-R$ to $R$. The function $f(z)$ has simple poles at $z = \pm 2i$ and a pole of multiplicity 2 at $z = \pm i$, of which $z = i$ and $z = 2i$ lie inside the closed contour $C$ (see Fig 1.2). Hence by Cauchy's residue theorem we obtain*

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i[Res(f;i) + Res(f;2i)]. \tag{6}$$

*Now*

$$\begin{aligned} Res(f;i) &= \lim_{z\to i} \frac{d}{dz}[(z-i)^2 f(z)] \\ &= \lim_{z\to i} \frac{d}{dz}\left[\frac{1}{(z+i)^2(z^2+4)}\right] \\ &= \lim_{z\to i} \frac{-2z(z+i)-2(z^2+4)}{(z+i)^3(z^2+4)^2} = -\frac{i}{36}. \end{aligned}$$

$$\begin{aligned} Res(f;2i) &= \lim_{z\to 2i}(z-2i)f(z) \\ &= \lim_{z\to 2i} \frac{1}{(z+2i)(z^2+1)^2} = -\frac{i}{36}. \end{aligned}$$

*Therefore from (6) we obtain*

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = \pi/9. \tag{7}$$

*Now on $C_R$,*

$$\begin{aligned} |\ f(z)\ | &= \frac{1}{|\ z^2+1\ |^2|\ z^2+4\ |} \le \frac{1}{(|\ z\ |^2-1)^2(|\ z\ |^2-4)} \\ &= \frac{1}{(R^2-1)^2(R^2-4)} = \frac{1}{R^6}\frac{1}{(1-\frac{1}{R^2})^2(1-\frac{4}{R^2})}. \end{aligned}$$

*Now applying ML-formula we obtain*

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{1}{R^6} \frac{1}{(1-\frac{1}{R^2})^2(1-\frac{4}{R^2})} \cdot \pi R \ \to 0 \ as \ R \to \infty$$

$$i.e. \ \ \lim_{R\to\infty} \int_{C_R} f(z)dz \ = 0.$$

*Therefore, proceeding to the limit as $R \to \infty$ we obtain from (7)*

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} \ = \pi/9$$

$$i.e. \ \ \int_0^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} \ = \pi/18.$$

*This completes the solution.*

**Note 1.** *It is not possible to obtain the integral of $f(x)$ over $[0, \infty)$ using the semi-circle $C_R$ if $f$ is not an even function.*

**Note 2.** *The technique of Examples 3 and 4 can be adopted to evaluate integrals of the form*

$$I \ = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx,$$

*where $p(x)$ and $q(x)$ are polynomials such that*

*(i) $q(x) \neq 0$ for $x \in \mathbb{R}$;*

*(ii) $p(x)$ and $q(x)$ have real coefficients;*

*(iii) deg $q(x) \ \geq$ deg $p(x) + 2$.*



**Integrals involving functions with infinitely many poles**

**Example 5.** *Evaluate* $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$. $(0 < a < 1)$.

**Solution.** *We consider*

$$\int_{\Gamma} \frac{e^{az}}{1+e^z} dz \ = \int_{\Gamma} f(z)dz, \ say,$$

*where $\Gamma$ is the rectangle $ABCD$ with vertices at $A(R, 0)$, $B(R, 2\pi)$, $C(-R, 2\pi)$, $D(-R, 0)$, $R$ being positive (see Fig 1.3). Therefore*

$$\int_{\Gamma} f(z)dz \ = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz. \tag{8}$$

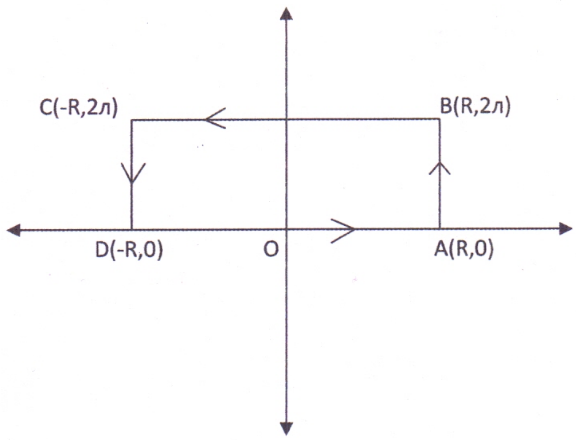


Fig. 3:

*The function $f(z)$ has simple poles at the points where*

$$1 + e^z = 0$$

$$i.e.\ e^z = -1 = e^{(2n+1)\pi i},\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

$$i.e.\ z = (2n+1)\pi i,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

*We see that among all these poles only $z = \pi i$ lie inside the contour $\Gamma$. Now*

$$\begin{aligned} Res(f;\pi i) &= \left[ \frac{e^{az}}{\frac{d}{dz}(1+e^z)} \right]_{z=\pi i} \\ &= \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}. \end{aligned}$$

*Therefore by Cauchy's residue theorem we have*

$$\int_{\Gamma} f(z)dz = -2\pi i e^{a\pi i}. \tag{9}$$

*On $AB$, $z = R + iy$, $0 \le y \le 2\pi$. Hence*

$$\begin{aligned} |\, f(z)\,| &= |\, \frac{e^{az}}{1+e^z} \,| = |\, \frac{e^{a(R+iy)}}{1+e^{R+iy}} \,| \\ &= |\, \frac{e^{aR}e^{aiy}}{1+e^R e^{iy}} \,| \le \frac{e^{aR}}{e^R - 1}. \end{aligned}$$

*Therefore by ML-formula we obtain*

$$|\int_{AB} f(z)dz\,| \le \frac{e^{aR}}{e^R-1} \cdot 2\pi \to 0 \ as\ R \to \infty.\ [since\ 0 < a < 1].$$

*Again on CD,* $z = -R + iy,\ 2\pi \geq y \geq 0$. *Hence*

$$\begin{aligned} |\ f(z)\ | &= \left|\ \frac{e^{az}}{1+e^{z}}\ \right| = \left|\ \frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\ \right| \\ &= \left|\ \frac{e^{-aR}e^{aiy}}{1+e^{-R}e^{iy}}\ \right| \leq \frac{e^{-aR}}{1-e^{-R}}. \end{aligned}$$

*Therefore by ML-formula we obtain*

$$\left|\ \int_{CD} f(z)dz\ \right| \leq \frac{e^{-aR}}{1-e^{-R}} \cdot 2\pi \ \to 0 \text{ as } R \to \infty. \ [\text{since } 0 < a < 1].$$

*Letting* $R \to \infty$ *we get from (8) and (9)*

$$\begin{aligned} -2\pi i e^{a\pi i} &= \int_{\infty}^{-\infty} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}dx \\ &= -\int_{-\infty}^{\infty} \frac{e^{ax}e^{2\pi ai}}{1+e^{x}e^{2\pi i}}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}dx \\ &= (1-e^{2\pi ai})\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}dx \\ \text{i.e.} \quad \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}dx &= -\frac{2\pi i e^{a\pi i}}{1-e^{2\pi ai}} \\ &= \frac{2\pi i}{e^{\pi ai}-e^{-\pi ai}} = \frac{\pi}{\sin\ a\pi}. \end{aligned}$$

*This completes the solution.*